Achievability of the Cramér-Rao Lower Bound in Astrometry

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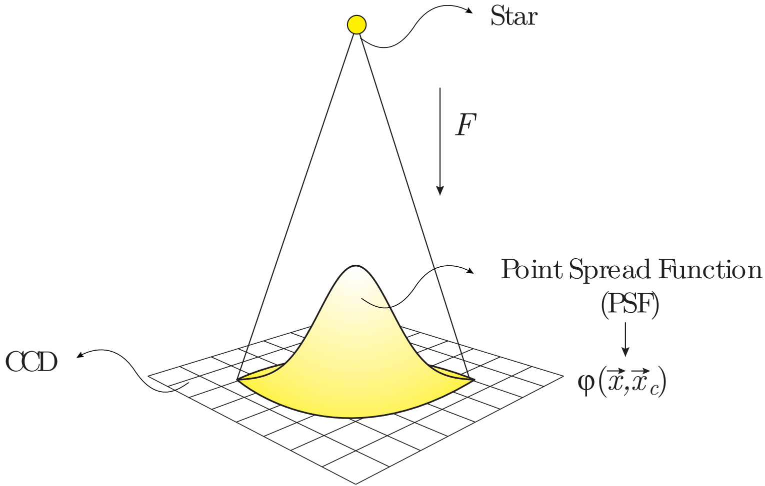
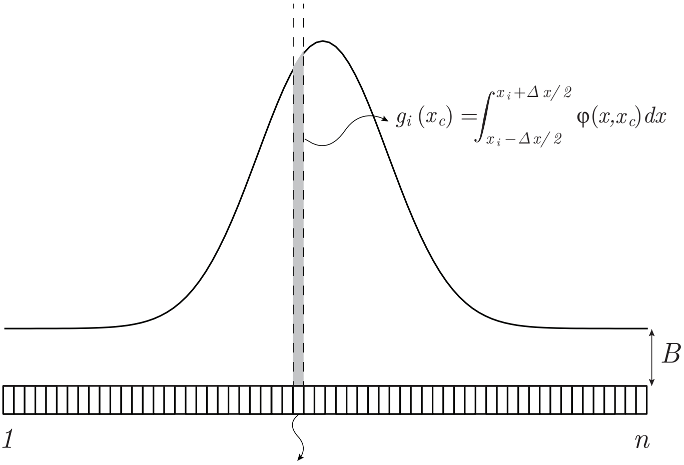
Motivation and Problem Setting (1)

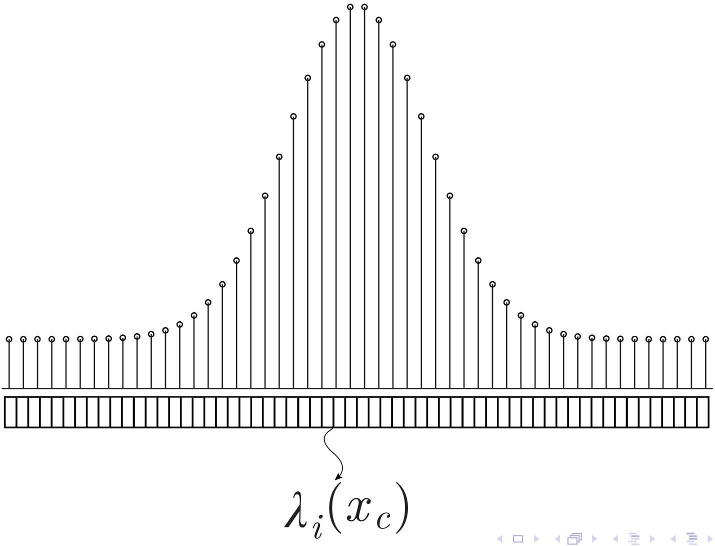


Motivation and Problem Setting (2)

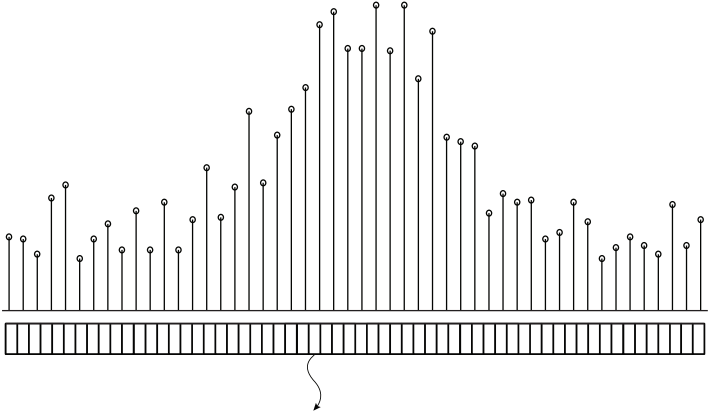


$$\lambda_i(x_c) = F \cdot g_i(x_c) + B$$

Motivation and Problem Setting (3)



Motivation and Problem Setting (4)



$$I_i \sim Poisson(\lambda_i(x_c))$$

Previous Work

Given a vector of observation $\overrightarrow{I} = (I_1, \ldots, I_n)$, what we want is a function (estimator) $\hat{x}_c(\overrightarrow{I})$ such that $\mathbb{E}\{\hat{x}_c(\overrightarrow{I})\} - x_c = 0$ and $Var(\hat{x}_c(\overrightarrow{I}))$ were minimal

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Mendez, Silva and Lobos (2013)

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where

$$\mathcal{I}_{x_c}(n) = \sum_{i=1}^{n} \frac{\left(F\frac{dg_i(x_c)}{dx_c}\right)^2}{Fg_i(x_c) + B} \tag{2}$$

New questions

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Proposition 1

If $\hat{x}_c$ corresponds to an unbiased estimator of x_c, then:

$$Var(\hat{x}_c) > \sigma^2_{CR} \tag{3}$$

The bound is unachievable

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Problem

$\hat{x}_{c_{LS}}$ does not have a closed form, then its variance cannot be compared with CRLB analytically

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Proposition 2

$$bias(\hat{x}_{c_{LS}}) \approx 0 \quad (6)$$

$$Var(\hat{x}_{c_{LS}}) \approx \frac{\mathbb{E}_{I_1,\ldots,I_n}\{J'(x_c\ ;\ \overrightarrow{I})^2\}}{\mathbb{E}_{I_1,\ldots,I_n}\{J''(x_c\ ;\ \overrightarrow{I})\}^2} \quad (7)$$

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$$= \frac{\sum_{i=1}^{n}\left(\frac{dg_i}{dx_c}\right)^2 (Fg_i + B)}{F^2\left(\sum_{i=1}^{n}\left(\frac{dg_i}{dx_c}\right)^2\right)^2} \quad (8)$$

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$$= \sigma_{LS}^2 \tag{9}$$

Least-Square estimator's variance analysis (2)

Proposition 3 (Two interesting regimes)

- If $F \cdot g_i \ll B$ (*Weak Source*), then

$$\frac{\sigma_{LS}^2}{\sigma_{CR}^2} \approx 1 \tag{10}$$

Least-Square estimator's variance analysis (2)

Proposition 3 (Two interesting regimes)

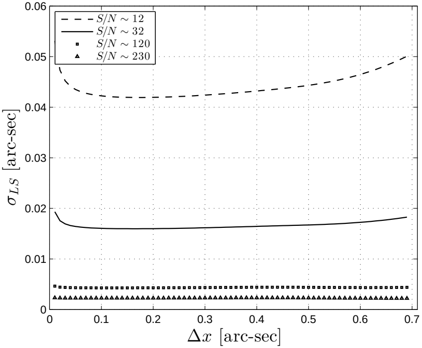
- If $F \cdot g_i \ll B$ (*Weak Source*), then

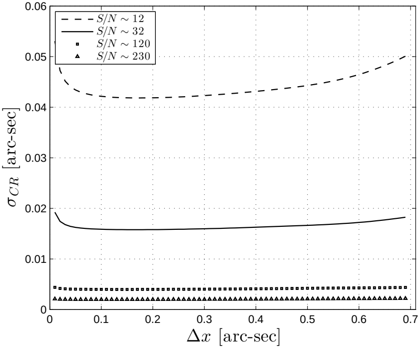
$$\frac{\sigma_{LS}^2}{\sigma_{CR}^2} \approx 1 \tag{10}$$

- If $B \ll F \cdot g_i$ (*Strong Source*) and $\Delta x / \sigma \ll 1$, then

$$\frac{\sigma_{LS}^2}{\sigma_{CR}^2} \approx \frac{8}{3\sqrt{3}} \approx 1.54 \tag{11}$$

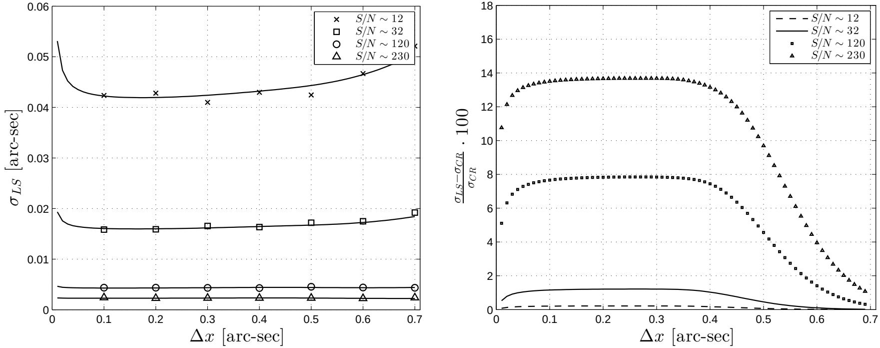
Experimental Results





LS standard deviation and root of the Cramér-Rao lower bound in arcseconds as a function of pixel size Δx in arcseconds ($FWHM = 1''$)

Experimental Results (2)



LS standard deviation and simulations as functions of pixel size Δx in arcseconds considering different regimes. Relative error between LS standard deviation and the root of Cramér-Rao lower bound, $\frac{\sigma_{LS}-\sigma_{CR}}{\sigma_{CR}} \cdot 100$, as a function of pixel size Δx in arcseconds.

Conclusions and Future Work

- There is a performance gap that can be studied with other estimators (WLS, ML)

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- An analytical expression of the variance of the LS estimator can be used as a design criterion
- Considering a priori knowledge can be useful in order to generate new estimators with a lower variance (more precision)

References

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