Papapetrou's equations of motion for an extended test body

W. Almonacid-Guerrero[1], Leonardo Castañeda[2]

Observatorio Astronómico Nacional, Universidad Nacional de Colombia

Abstract

We use Dixon's general equations of motion for extended bodies to compute the Papapetrou's equations for an extended test body in Schwarzschild space-time. We incorporate the force and torque terms which involve multipolar moments. The Corinaldesi-Papapetrou spin supplementary condition is introduced to obtain the equations of motion in the rest frame of the Schwarzschild field.

Introduction

The general equations of motion for an extended body in a given gravitational background were obtained by Dixon [1] in multipole approximation of the body structure for any order. The method involves the definition of a world-tube enclosing the entire body, an appropriate foliation of the spacetime and a convenient worldline representing the center of mass (CM), around which the multipole expansion are performed. The set of equations for extended bodies is

$$F^{\nu} \equiv \frac{\delta p^{\nu}}{\delta s} - \frac{1}{2} S^{\kappa\lambda} v^{\mu} R_{\kappa\lambda\mu}{}^{\nu}, \tag{1}$$

$$L^{\kappa\lambda} \equiv \frac{\delta S^{\kappa\lambda}}{\delta s} - 2p^{[\kappa} v^{\lambda]}, \tag{2}$$

where $p^{\nu} = M u^{\nu}$ is the total four-momentum, $S^{\kappa\lambda}$ is the spin tensor, F^{ν} and $L^{\kappa\lambda}$ are the force and torque linked with the structure of the body beyond the quadrupole terms.

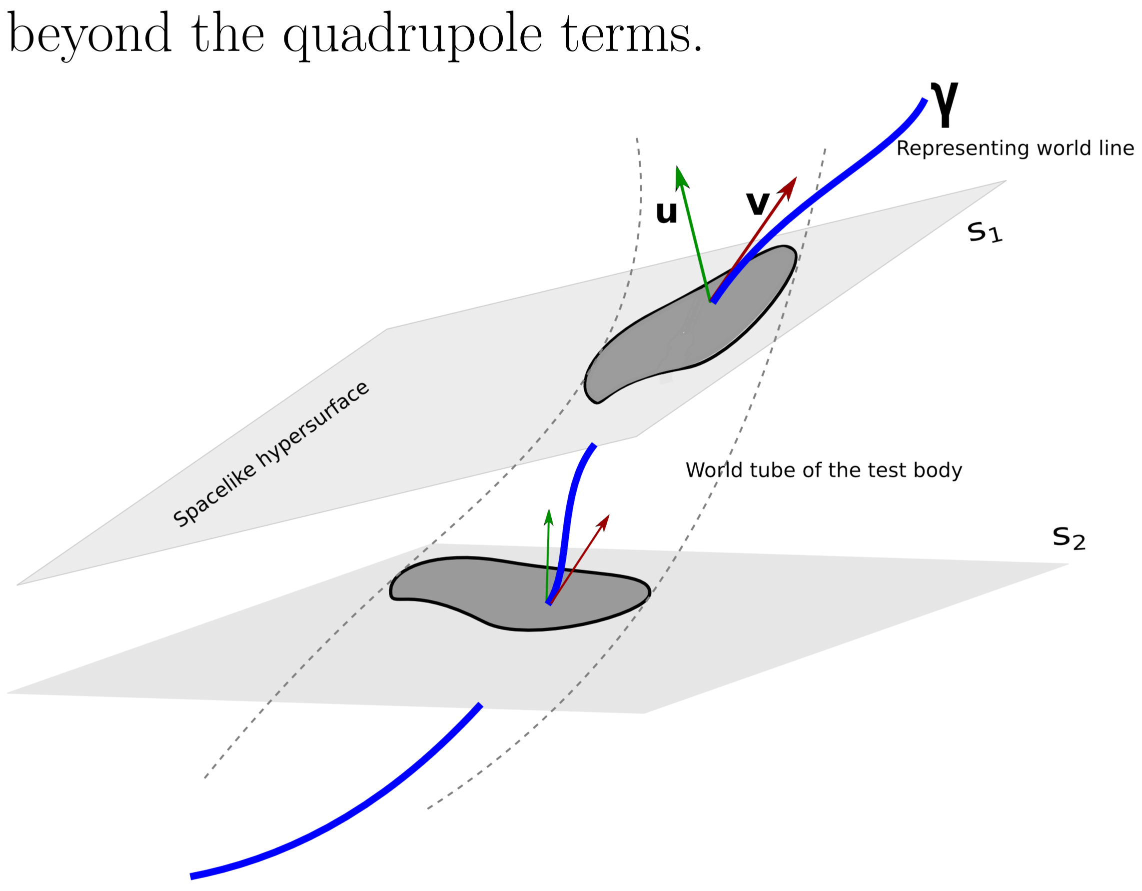

Figure 1: The timelike world line enclosed by a world tube. The spacelike hypersurfaces are orthogonal to $u^{\mu} = (v^0, \mathbf{u})$ and $v^{\mu} = (v^0, \mathbf{v})$ is tangent to the world line.

Even before Dixon's work, Papapetrou [2] derived the equations of motion of spinning test particles, which are the starting point for the analysis of the precession of gyroscopes. We consider these equations as a particular case of the equations (1-2), and compute the equations of motion for an extended test body immersed in Schwarzschild spacetime. We impose the Corinaldesi-Papapetrou spin supplementary condition which specifies the line γ representing the motion of the center of mass.

[1] waalmonacidg@unal.edu.co

[2] lcastanedac@unal.edu.co

Notation and conventions

The Schwarzschild metric in Cartesian coordinates (t, x^i) is written as

$$g_{00} = -e^{\mu}, \quad g_{ij} = \delta_{ij} - \frac{(1-e^{-\mu})}{r^2} x^i x^j, \tag{3}$$

with $e^{\mu} = -(1 - 2r_0/r)$, and $r_0 = \frac{Gm}{c^2}$ being the gravitational radius of the central body.

Usually to equations (1-2), supplementary conditions are added which single out the worldline of the CM. We apply the Corinaldesi-Papapetrou condition which holds in the rest-frame of the Schwarzschild field. As a consequence of this choosing the independent components of the spin tensor reduces to three, thus

$$S^k \equiv \frac{1}{2} \epsilon_{ijm} \delta^{km} S^{ij}, \tag{4}$$

Equations of motion

Let λ^{α} to be a four-vector defined by

$$\lambda^0 = c\frac{2r_0}{r^3} e^{-\mu} (\mathbf{r}\cdot\mathbf{v})\dot{t},$$
$$\boldsymbol{\lambda} = \frac{r_0}{r^3}\left[c^2 e^{\mu}\dot{t}^2 + 2|\mathbf{v}|^2 - \frac{1}{r^2}\left(2+e^{-\mu}\right)(\mathbf{r}\cdot\mathbf{v})^2\right]\mathbf{r}, \tag{5}$$

and let the torque terms, related to high multipolar structure of the test body,

$$\boldsymbol{\tau} = \frac{1}{2}\epsilon_{ijm}\left(\frac{1}{2}L^{ij} + \frac{1}{v^0}L^{0[i}v^{j]}\right)\delta^{km}, \tag{6}$$

$$\varsigma^i = \frac{\delta}{\delta s}\left(\frac{1}{v^0}L^{0i}\right). \tag{7}$$

Then, the nongeodesic equations of motion can be expressed as

CM equation

$$\frac{d}{ds}(M_* v^0) + M_*\lambda^0 - F^0 = 0 \tag{8}$$

$$\frac{d}{ds}(M_*\mathbf{v}) + M_*\boldsymbol{\lambda} + \frac{3r_0}{r^5}\left[(\mathbf{S}\cdot\mathbf{r})(\mathbf{r}\times\mathbf{v}) + e^{-\mu}(\mathbf{r}\cdot\mathbf{v})(\mathbf{r}\times\mathbf{S})\right] + \mathbf{F} - \frac{r_0}{r^3}e^{-\mu}(\mathbf{r}\times\boldsymbol{\tau}) - \boldsymbol{\varsigma} = 0. \tag{9}$$

Spin equation

$$\dot{\mathbf{S}} - \frac{r_0}{r^3}\left[2e^{-\mu}(\mathbf{r}\cdot\mathbf{v})\mathbf{S} + 2(\mathbf{r}\cdot\mathbf{S})\mathbf{v} - e^{-\mu}(\mathbf{v}\cdot\mathbf{S})\mathbf{r} - \frac{1}{r^2}\left(2+e^{-\mu}\right)(\mathbf{r}\cdot\mathbf{v})(\mathbf{r}\cdot\mathbf{S})\mathbf{r}\right] - \boldsymbol{\tau} = 0 \tag{10}$$

Differentiation with respect to the parameter s are denoted with a dot.

Effective Mass

In the equations (8-10), M_* represents an effective mass associated with the mass of the body plus an energetic component which results from the interaction between the multipolar structure of the body and the spacetime curvature.

$$M_* = M + M_s + M_L, \tag{11}$$

where M is a positive scalar comming from the four-momentum definition. In general it is not constant and its variation depends on the high multipolar structure. Hence, dM/ds vanishes whenever one neglects the force and torque which arise from the multipole moments of the body. M_s assumes the characteristic form of a spin-orbit interaction energy and M_L represents the energy associated with the interaction between the structure of the test body and the gravitational fields. They are written as

$$\boxed{M_s = \frac{r_0}{Mr^3}e^{-\mu}(\mathbf{r}\times\mathbf{p})\cdot\mathbf{S}}, \quad \text{and} \quad \boxed{M_L = \frac{u_{\sigma}}{v^0}L^{0\sigma}}. \tag{12}$$

Conclusion

We have examined the Papapetrou's equations of motion for an extended body with arbitrary multipolar structure. The highlighted terms in the equations (8-10) are new contributions to the motion of the spinning test body associated with the quadrupole and higher multipolar structure of the body, which depends on its stress-energy tensor and the gravitational fields. In the dipole approximation this equations reduce to the classical Papapetrou's equations. Also, we present an additional contribution to the mass of the body in (12).

References

[1] W. G. Dixon. *Dynamics of extended bodies in general relativity I.* Proc. Roy. Soc. Lond. A. 314 (1970)

[2] A. Papapetrou. *Spinning test particles in general relativity I.* Proc. Roy. Soc. Lond. A. 209 (1951)

[3] W. Almonacid; L. Castañeda. *Equations of motion of extended test bodies in static and isotropic metrics.* In preparation.